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Dynamics of Sine-Gordon Solitons

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After reviewing a few physical examples in which the sine-Gordon equation arises as the governing dynamical equation, we discuss various solutions exhibiting multisoliton dynamics. Interaction of solitons and the corresponding velocity-dependent interaction potentials are derived and discussed. Numerical experiments are carried out in order to study kink dynamics in an inhomogeneous medium. Finally, we introduce two kinds of generalized sine-Gordon equations and discuss their properties.

1. INTRODUCTION

There is a hope that some kind of nonlinear field theory will eventually lead to an explanation of elementary particles like quarks and leptons, their interactions, and even mass spectrum (Skyrme, 1988). The invention of skyrmions or chiral solitons (Skyrme, 1961, 1962) is probably the most successful step toward this grand goal. In this model, baryons are recognized as solitons of a nonlinear chiral field of mesons. The conserved baryon number is constructed via a topological current with quantized charges. Various properties of nucleons (like mass, isospin, magnetic moments, etc.) are predicted to some 20% accuracy.

As many fundamental properties of solitons in nonlinear field theories are still far from being well understood, any analytical or numerical attempt to highlight these properties even in simpler models could be very fruitful.

One of the best-studied equations with well-defined multisoliton solutions is the sine-Gordon equation. This equation has a long history, starting from studies about curves and surfaces in differential geometry (Eisenhart,

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1960; see also the interesting article by McLachlan, 1994). A classification of the solutions of this equation was given by Steuerwald (1936).

Bäcklund transformations for generating, multisoliton solutions of nonlinear equations appeared in 1876 (Bäcklund, 1876). Despite its simple-looking appearance, the sine-Gordon equation contains a variety of solutions and a rich dynamics. Among these interesting features, particle aspects of kink solutions and their interactions (Bowtell and Stuart, 1977; Riazi, 1993) can be mentioned. Although the interaction between kinks is quite complex and velocity dependent, it can be studied very easily because exact analytical expressions exist which represent the dynamics of many kinks in interaction (see Section 5). This situation may be compared to the complexities involved in the few-body problem of classical mechanics. The dynamics of a system of many interacting kinks from the viewpoint of statistical mechanics has been studied (see, e.g., Babelon, 1993; Marchesoni, 1992). In this paper, we first review two physical examples, long Josephson junctions and self-induced transparency, which lead to the sine-Gordon equation (Sections 2 and 3, respectively). In Section 4 we derive a few elementary solutions which describe single- and double-kink configurations. Topological and dynamical properties of kinks and interkink interactions are derived and discussed in this section. In particular, the velocity dependence of the corresponding potentials are studied.

Bäcklund transformations for generating multisoliton solutions of the sine-Gordon equation are reviewed and applied in Section 5. Numerical experiments are presented in Section 6. These calculations concern the dynamics of kinks in an inhomogeneous background. Inhomogeneity can be introduced into sine-Gordon equation at least in three ways. Two types of inhomogeneity are studied, and a classical mechanical description for inhomogeneity of each kind is presented. Inhomogeneity of the first kind is physically related to a long Josephson junction with a spatially varying dielectric constant. Finally, in Section 7, we introduce two types of modified sine-Gordon equations with nondegenerate solitary wave masses and study their interactions.

2. JOSEPHSON TRANSMISSION LINES

Superconducting Josephson junctions (Josephson, 1962) exhibit soliton behavior which can be described by a sine-Gordon equation (see, e.g., Pederson, 1986). Solitons are quanta of magnetic flux in this case. Their production, transmission, and storage as stable objects is quite fissible, and therefore very important in information processing systems.

The tunneling effect of Cooper pairs across a thin insulator between two superconductors was predicted by Josephson (1962). If the common macroscopic wave function of all the electron pairs is written as

$$\Psi = \sqrt{R}e^{i\varphi} \tag{1}$$

the two superconductors will naturally have independent wave functions Ψ_1 and Ψ_2 with uncorrelated phases φ_1 and φ_2, unless the two superconductors are set near enough to each other (say less than about 30 Å). The phases then become correlated because of Cooper pair penetration through the insulator barrier. The wave functions Ψ_1 and Ψ_2 satisfy two coupled linear Schrödinger equations (Feynman *et al.*, 1965)

$$i\hbar \frac{\partial \Psi_1}{\partial t} = E_1\Psi_1 + k\Psi_2 \tag{2}$$

$$i\hbar \frac{\partial \Psi_2}{\partial t} = E_2\Psi_2 + k\Psi_1 \tag{3}$$

where E_1 and E_2 are the ground-state energies of electrons in the two superconductors. Here, we have assumed that the two superconductors are similar. k is a real coupling constant which depends on the characteristics of the junction. Obviously, $k \to 0$ as $d \to \infty$, where d is the barrier thickness. When a static potential difference V is maintained between the two superconductors, an energy shift $E_1 - E_2 = 2eV$ is developed. We can arbitrarily choose the reference energy at $E = (E_1 + E_2)/2 = 0$, and therefore $E_1 = eV$ and $E_2 = -eV$. Equations (2) and (3) then become

$$i\hbar \frac{\partial \Psi_1}{\partial t} = eV\Psi_1 + k\Psi_2 \tag{4}$$

$$i\hbar \frac{\partial \Psi_2}{\partial t} = -eV\Psi_2 + k\Psi_1 \tag{5}$$

Using the expression $\Psi_1 = \sqrt{R_1}e^{i\varphi_1}$ and $\Psi_2 = \sqrt{R_2}e^{i\varphi_2}$ in these equations and separating the real and imaginary parts, we obtain

$$\hbar \frac{\partial R_1}{\partial t} = -2k\sqrt{R_1R_2}\sin\varphi \tag{6}$$

$$\hbar \frac{\partial R_2}{\partial t} = +2k\sqrt{R_1R_2}\sin\varphi \tag{7}$$

$$\hbar \frac{\partial \varphi_1}{\partial t} = k\sqrt{R_2/R_1}\cos\varphi - eV \tag{8}$$

$$\hbar \frac{\partial \varphi_2}{\partial t} = k\sqrt{R_1/R_2} \cos \varphi + eV \tag{9}$$

in which $\varphi = \varphi_2 - \varphi_1$ is the phase difference between the two wave functions.

Let us define the quantities $J_1 \equiv \partial R_1/\partial t$ and $J_2 \equiv \partial R_2/\partial t$, where R_1 and R_2 represent electron pair densities which deviate only slightly from their equilibrium values R_0. We therefore have $R_1 \simeq R_2 \simeq R_0$, and $(2k/\hbar)\sqrt{R_1 R_2} \simeq 2kR_0/\hbar \equiv J_0$, and therefore

$$J \simeq J_0 \sin \varphi \tag{10}$$

according to (6) or (7).

Equations (8) and (9) therefore yield

$$\hbar \frac{\partial \varphi}{\partial t} = 2eV \tag{11}$$

We can write equation (11) in the form

$$\frac{d\Phi}{dt} = V \tag{12}$$

where Φ has the dimensions of magnetic flux, and is *defined* according to

$$\varphi = 2\pi \frac{\Phi}{\Phi_0} \tag{13}$$

in which $\Phi_0 \equiv h/2e$ is the quantum of magnetic flux. From (10) and (13) we have

$$\Phi = \frac{\Phi_0}{2\pi} \sin^{-1} \frac{J}{J_0} \tag{14}$$

If $V = 0$, then (13) implies $\Phi =$ const, which is in general nonvanishing. This leads to a finite current density J even in the absence of an applied voltage. The effect is known as the dc Josephson effect. If $V = V_0 = const$, $\Phi = V_0 t + \Phi_1$ and (15) yields an alternating current density (ac Josephson effect)

$$J = \frac{2\pi J_0}{\Phi_0} \sin \frac{2\pi}{\Phi_0} (V_0 t + \Phi_1) \tag{15}$$

Therefore, an alternating current density develops with an angular frequency

$$\omega_J = \frac{2\pi V_0}{\Phi_0} = \frac{2eV_0}{\hbar} \tag{16}$$

This frequency is of the order of a few hundred MHz per μV voltage difference.

We now turn to a long Josephson junction, which consists of two relatively long strips of superconducting materials separated by a very thin dielectric of thickness d. It can be shown that a length element dx of this device is electrically equivalent to an electrical circuit with capacitance per unit length

$$C = \frac{K\varepsilon_0 a}{d} \tag{17}$$

in which K is the dielectric constant of the dielectric, $\varepsilon_0 = 8.854 \times 10^{-12}$ $C^2/N \cdot m^2$, and a is the width of the superconducting strip. Inductance per unit length is

$$L = \mu_0 \frac{2\lambda_L + d}{a} \tag{18}$$

where $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}, and λ_L is the penetration depth of the superconductors.

From basic circuit theory, the following equations result:

$$\frac{\partial V}{\partial \chi} = -L\frac{\partial I}{\partial t} \tag{19}$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} - \frac{2\pi J_0}{\Phi_0}\sin 2\pi\frac{\Phi}{\Phi_0} \tag{20}$$

$$\frac{\partial \Phi}{\partial t} = V \tag{21}$$

These equations can be easily combined to yield the following sine-Gordon equation for the phase difference:

$$\frac{\partial^2 \varphi}{\partial t^2} - c_J^2\frac{\partial^2 \varphi}{\partial x^2} + \omega_p^2 \sin\varphi = 0 \tag{22}$$

in which

$$c_J = \frac{1}{\sqrt{LC}} \qquad \textit{and} \qquad \omega_p = \sqrt{\frac{2\pi J_0}{\Phi_0 c}} \tag{23}$$

and (14) has been used. Note that c_J/ω_p has dimensions of length. It describes a length scale (called the Josephson penetration length), which determines whether a Josephson junction is “long” or not. Equation (22) can obviously have the kink solution (86) (see Section 4.1), in which $b = 1$, $a = \omega_p^2/c_J^2$, and $c \to c_J$. The corresponding voltage V and current I can then be easily calculated using equations (20) and (21). The kink (antikink) describes a pulse of 2π (-2π) phase difference, corresponding to a quantum of magnetic flux accompanied by a voltage and current pulse. The kink (antikink) is thus called a fluxon (antifluxon) in this case.

Any spatial variation in the dielectric constant K results in a position-dependent c_J. This in turn affects the propagation of kinks in the junction. The situation can be approximated by the classical analog of a point particle moving in a (velocity-dependent) external potential. We will discuss this further in Section 6.

3. SELF-INDUCED TRANSPARENCY

3.1. Introduction

Interaction of an intense incident light with atoms can lead to interesting nonlinear phenomena. Strong resonant interaction occurs when the photon energy is very nearly equal to the energy difference between two atomic levels. We will closely follow Lamb (1980) in this section.

In order to simplify the analysis, we consider a two-level atom, and treat the electromagnetic field classically. We will observe that solitons may occur in such interactions.

Self-induced transparency is an effect which occurs when solitons form in the resonance interaction of radiation with atoms. If the two-level gas atoms are irradiated with an electromagnetic pulse of suitable envelope, despite resonant interaction, a more or less similar output pulse envelope is received. In such a case, attenuation due to scattering is almost suppressed.

From Maxwell's equations, the following wave equation is easily obtained for propagation of the electric field.

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 \mathbf{P}}{\partial t^2} \tag{24}$$

in which $\mathbf{P}$ is the polarization (electric dipole density) of the medium. The relation between $\mathbf{P}$ and $\mathbf{E}$ is in general nonlinear. This is the source of interesting nonlinear phenomena.

The electric field is assumed to be in the form of very short duration (picosecond) plane-polarized pulses, about 3–6 orders of magnitude longer than the period of the wave oscillations. The electric field can be written as

$$E(x, t) = \varepsilon(x, t) \cos[\mathrm{k}x - \omega t + \varphi(x, t)] \tag{25}$$

in which $\varepsilon(x, t)$ is the envelope and $\varphi(x, t)$ is the phase, both varying slowly with time and position:

$$\frac{\partial \ln \varepsilon}{\partial x} << k; \qquad \frac{\partial \ln \varepsilon}{\partial t} << \omega \tag{26}$$

Similar inequalities hold for φ. We will find approximate equations governing ε and φ, and will consider the interaction of stationary atoms with this applied electric field.

3.2. Quantum Mechanics

We consider a two-level (a and b) atom having an energy difference almost equal to the incident electromagnetic field frequency times the Planck constant, $E_a - E_b \simeq \hbar\omega_0$. The wave function of the atom can be written as a time-dependent linear combination of the two normalized wave functions $\psi_a(\mathbf{r})$ and $\psi_b(\mathbf{r})$

$$\psi(\mathbf{r}, t) = a(t)\psi_a(\mathbf{r}) + b(t)\psi_b(\mathbf{r}) \tag{27}$$

where $\int |\psi_{a,b}|^2 \, d^3x = 1$. The wave function $\psi(\mathbf{r}, t)$ is also normalized:

$$\begin{aligned}\int \psi^* \, \psi \, d^3x &= |a|^2 + |b|^2 + a^*b \int \psi_a^* \, \psi_b \, d^3x + ab^* \int \psi_b^* \psi_a \, d^3x \\ &= |a|^2 + |b|^2 = 1 \end{aligned} \tag{28}$$

in which the orthonormality of ψ_a and ψ_b have been used. If n_0 is the number of atoms per unit volume, the population difference n will be

$$n = n_0 \int (|a\psi_a|)^2 - |b\psi_b|^2) \, d^3x = n_0 \, (|a|^2 - |b|^2) \tag{29}$$

The wave function ψ satisfies the time-dependent Schrödinger equation

$$H\psi = i\hbar \frac{\partial \psi}{\partial t} \tag{30}$$

The Hamiltonian consists of H_0 (free atom Hamiltonian), $-\hbar^2/2m\nabla^2$ (center-of-mass kinetic energy), and $H_I = -\mathbf{d} \cdot \mathbf{E}$ (electric dipole interaction). $\mathbf{d}$ is the electric dipole moment

$$\mathbf{d} = -\, e\mathbf{r} \tag{31}$$

where $\mathbf{r}$ is an internal position vector for the electron. $\mathbf{d}$ is assumed to be parallel with $\mathbf{E}$ so that $H_I = -\mathbf{d} \cdot \mathbf{E}$. We obtain p via

$$p = \int \psi^* \, d\psi \, d^3x \tag{32}$$

The atom is assumed to have no permanent electric dipole moment, so that $\int \psi_a^* \mathbf{r} \psi_a \, d^3x = \int \psi_b^* \mathbf{r} \psi_b \, d^3x = 0$. Equation (32) then implies

$$p = p_0(a^*b + b^*a) \tag{33}$$

where

$$p_0 \equiv -e \int \psi_a^* r \psi_b \, d^3x = -e \int \psi_b^* r \psi_a \, d^3x \tag{34}$$

Multiplying (30) by ψ_a^* and integrating over all space and using the orthogonality relations for ψ_a and ψ_b, we obtain

$$\frac{d}{dt} a(t) + i\omega_a a(t) = -iVb(t) \tag{35}$$

and likewise

$$\frac{d}{dt} b(t) + i\omega_b b(t) = -iVa(t) \tag{36}$$

where $\omega_a = E_a/\hbar$, $\omega_b = E_b/\hbar$, and $V = -p_0 E(x, t)/\hbar$.

3.3. Derivation of Sine-Gordon Equation: Stationary Atoms

For a discussion about the interaction of the incident radiation with moving atoms, the reader is referred to Lamb (1980). Here, we assume that all atoms are stationary. The electric field at the position of an atom at x_0 is therefore

$$E(x_0, t) = \mathcal{E}(x, t) \cos [kx_0 - \omega t + \varphi(x_0, t)] \tag{37}$$

ε and φ are assumed to be very slowly varying functions of time and position, and the second harmonic generation is neglected. Let us set

$$a = iv_1 \exp\left[-i\omega_a\left(t - \frac{x_0}{c}\right)\right] \tag{38}$$

$$b = v_2 \exp\left[-i\omega_b\left(t - \frac{x_0}{c}\right)\right] \tag{39}$$

We thus obtain from (35)–(39)

$$v_{1t} = \frac{p_0\mathcal{E}}{2\hbar} e^{i\varphi} v_2 \tag{40}$$

$$v_{2t} = -\frac{p_0\mathcal{E}}{2\hbar} e^{-i\varphi} v_1 \tag{41}$$

The normalization condition now reads $|v_1|^2 + |v_2|^2 = 1$, and the normalized population density difference reads

$$\mathcal{N} = |v_1|^2 - |v_2|^2 \tag{42}$$

From (33), the polarization of an atom becomes

$$p = p_0[-iv_1^* v_2 e^{i\omega(t - x_0/c)} + C.C.] \tag{43}$$

If we define $\Phi \equiv kx_0 - \omega t + \varphi$, we obtain

$$p = p_0(\Xi \cos \Phi + \Pi \sin \Phi) \tag{44}$$

where the polarization envelope functions Ξ and Π are

$$\Xi(\Delta\omega, x, t) = i(v_1 v_2^* e^{-i\varphi} - v_1^* v_2 e^{i\varphi}) \tag{45}$$

$$\Pi(\Delta\omega, x, t) = -(v_1 v_2^* e^{-i\varphi} + v_1^* v_2 e^{i\varphi}) \tag{46}$$

The total polarization per unit volume is

$$P(x, t) = n_0 p(x, t) \tag{47}$$

We have

$$\frac{\partial^2 P}{\partial t^2} \simeq -\omega_0^2 P = -n_0\omega_0^2(\Xi \cos \Phi + \pi \sin \Phi) \tag{48}$$

in which we have neglected slower time variations of P.

Putting (37) and (48) into (24), and neglecting all terms quadratic in $(\partial/\partial t)\varphi$, $(\partial/\partial x)\varphi$, $(\partial/\partial x)\mathscr{E}$, and $(\partial/\partial t)\mathscr{E}$, we find (after equating the coefficients of $\sin \Phi$ and $\cos \Phi$)

$$\frac{\partial \mathscr{E}}{\partial t} + c\frac{\partial \mathscr{E}}{\partial x} = 2\pi n_0 \omega_0 p_0 \Pi(x, t) \tag{49}$$

$$\mathscr{E}\left(\frac{\partial \varphi}{\partial t} + c\frac{\partial \varphi}{\partial x}\right) = 2\pi n_0 \omega_0 p_0 \Xi(x, t) \tag{50}$$

These two equations can be combined into the following complex equation:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)(\mathscr{E} e^{i\varphi}) = 2\pi n_0 \omega_0 p_0 (\Pi + i\Xi) e^{i\varphi} \tag{51}$$

It can also be shown easily that

$$\frac{\partial \Pi}{\partial t} \equiv \Pi_t = \frac{p_0 \mathscr{E}}{\hbar}\mathcal{N} + \varphi_t \Xi \tag{52}$$

$$\mathcal{N}_t = -\frac{p_0 \mathscr{E}}{\hbar}\Pi \tag{53}$$

$$\Xi_t = -\varphi_t \Pi \tag{54}$$

These are special cases of Bloch's equations (Bloch, 1946), suitable for stationary atoms.

Let us use the change of variable

$$\mathscr{E} = \frac{p_0 \mathscr{E}}{\hbar} \tag{55}$$

$\mathscr{E}$ has the dimensions of frequency. The governing equations thus become

$$\frac{\partial \mathscr{E}}{\partial t} + c\,\frac{\partial \mathscr{E}}{\partial x} = \Omega^2 \Pi(x,\ t) \tag{56}$$

$$\frac{\partial \Pi}{\partial t} = \mathscr{E}\mathscr{N} \tag{57}$$

$$\frac{\partial \mathscr{N}}{\partial t} = -\mathscr{E}\Pi \tag{58}$$

in which

$$\Omega^2 = \frac{2\pi n_0 \omega_0 P_0^2}{\hbar} \tag{59}$$

Note that if $\Xi(x, 0) = \Xi_t(x, 0) = 0$, then Ξ remains constant and we have $\varphi_t = 0$. Equations (57) and (58) immediately yield

$$\mathscr{N}^2 + \Pi^2 = 1 \tag{60}$$

which enables us to write

$$\Pi = \pm \sin \sigma \tag{61}$$

$$\mathscr{N} = \pm \cos \sigma \tag{62}$$

From (57) or (58) we obtain

$$\mathscr{E} = \frac{\partial \sigma}{\partial t} \tag{63}$$

We can therefore write $\sigma(x, t) = \int_{-\infty}^{t} dt'\ \mathscr{E}(x, t')$. Obviously $\sigma(x, -\infty) = 0$, which yields $\Pi(t = -\infty) = 0$ and $\mathscr{N}(t = -\infty) = \pm 1$. $\mathscr{N} = +1$ corresponds to the initially inverted atomic population, while $\mathscr{N} = -1$ corresponds to all atoms being in the lower level.

Let us use the coordinate transformations

$$\xi = \frac{\Omega x}{c} \tag{64}$$

$$\tau = \Omega\left(t - \frac{x}{c}\right) \tag{65}$$

This transformation implies

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = \Omega \frac{\partial}{\partial \tau} \tag{66}$$

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \lambda}{\partial x}\frac{\partial}{\partial r} = \frac{\Omega}{c}\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \tau}\right) \tag{67}$$

(56) then takes the form

$$\frac{\partial^2 \sigma}{\partial \xi\, \partial r} = \pm \sin \sigma \tag{68}$$

which is one of the common forms of the sine-Gordon equation (see Section 5).

3.4. Physical Interpretation

The single-soliton (kink, antikink) solution to this equation is (see the next section)

$$\sigma(\xi, \tau) = 4 \tan^{-1}[\exp(\alpha\tau \pm \xi/\alpha)] \tag{69}$$

In terms of the dimensionless electric field envelope

$$u = \frac{\mathscr{E}}{\Omega} \tag{70}$$

we have

$$\frac{\partial u}{\partial \varepsilon} = \pm \sin \sigma \tag{71}$$

and

$$\int_{-\infty}^{+\infty} d\tau u(\xi, \tau) = \int_{-\infty}^{+\infty} \frac{\mathscr{E}}{\Omega}\, d\tau = \int_{-\infty}^{+\infty} \frac{1}{\Omega}\frac{\partial \sigma}{\partial t}\, d\tau = \int_{-\infty}^{+\infty} \frac{\partial \sigma}{\partial \tau}\, d\tau = 2\pi \tag{72}$$

for the kink solution. If the area under the pulse (kink) is slightly greater than 2π, we have

$$\frac{\partial u}{\partial \xi} = \pm \sin(2\pi + \varepsilon) \simeq \pm \varepsilon \tag{73}$$

For the plus sign, $\partial u/\partial \xi > 0$, which forces $\mathscr{E}$ to increase further. For the minus sign, $\partial u/\partial\xi < 0$ for $\varepsilon > 0$ and $\partial u/\partial\xi > 0$ for $\varepsilon < 0$, which shows that the area converges toward 2π. We can conclude that the pulse is stable only in the latter case.

We can interpret the occurrence of the soliton behavior in the following way. The leading part of $\mathscr{E}$ causes the atoms to be excited and the population to be inverted. The trailing part of the pulse causes stimulated emission and the attenuated pulse gets amplified and is thus recovered. The process occurs only if the pulse duration is short enough, during which atoms do not undergo collisions and thus their coherency with the incident light is not lost. The light intensity should also be strong enough so that the majority of the atoms are involved in the process, otherwise attenuation will result. From (63), (69), and (70) we have

$$\mathscr{E} = \frac{p_0}{\hbar}\mathscr{E} = \Omega\frac{\partial\sigma}{\partial\tau} = \frac{2}{\tau_p}\,\text{sech}\left[\frac{t - x/v}{\tau_p}\right] \tag{74}$$

where $\tau_p = (\alpha\Omega)^{-1}$ and

$$\frac{1}{v} = \frac{1}{c}\left(1 + \frac{1}{\alpha'^2}\right) \tag{75}$$

The pulse velocity can be a few orders of magnitude smaller than the phase velocity of light waves in the medium, depending on the value of α.

For further information about the numerical and experimental results, the reader can refer to Patel and Slusher (1968), McCall and Hahn (1969), and Gibbs and Slusher (1972).

4. SOLITONS OF SINE-GORDON EQUATION

4.1. Separation of Variables

The standard form of the sine-Gordon equation is

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = \alpha \sin b\varphi \tag{76}$$

in which a and b are constants assumed to have the same sign. By using the change of variables

$$u = \sqrt{ab}x; \qquad v = \sqrt{ab}ct; \qquad \sigma = b\varphi \tag{77}$$

equation (76) becomes

$$\sigma_{uu} - \sigma_{vv} = \sin\sigma \tag{78}$$

Multisoliton solutions of this equation can be obtained systematically by applying Bäcklund transformations (Section 5). In this section, we will obtain a restricted class of single- and double-kink solutions via an elementary separation-of-variable method (Lamb, 1980). Using the ansatz

$$\sigma(u, v) = 4 \tan^{-1} \frac{U(u)}{V(v)} \tag{79}$$

and the trigonometric identity

$$\sin \sigma = \frac{4 \tan(\sigma/4)[1 - \tan^2(\sigma/4)]}{[1 + \tan^2(\sigma/4)]^2} \tag{80}$$

we obtain

$$(U^2 + V^2)\left(\frac{U''}{U} + \frac{V''}{V}\right) - 2(U')^2 - 2(V')^2 = V^2 - U^2 \tag{81}$$

in which the primes indicate differentiation of the functions U and V with respect to their arguments. By differentiating (81) once with respect to u and once with respect to v, we can separate this equation into

$$\frac{1}{UU'}\left(\frac{U''}{U}\right)' = -\frac{1}{VV'}\left(\frac{V''}{V}\right)' = -4\,k^2 \tag{82}$$

These equations can now be easily twice integrated to yield

$$(U')^2 = -k^2U^4 + m^2U^2 + n^2 \tag{83}$$

$$(V')^2 = k^2V^4 + (m^2 - 1)V^2 - n^2 \tag{84}$$

in which m and n are integration constants. Solutions of these equations involve, in general, elliptic functions (Steuerwald, 1936).

There are a few special cases which can yield simple, yet important soliton solutions.

4.2. Single-Soliton (Kink) Solutions

Let $k = 0$, $m > 1$, and $n = 0$ in (83) and (84). These equations can immediately be integrated to yield $U(u) = \gamma_1 \exp(\pm mu)$ and $V(v) = \gamma_2 \exp[\pm(m^2 - 1)^{1/2}\, v]$, in which γ_1 and γ_2 are constants of integration. Therefore

$$\sigma(u, v) = 4 \tan^{-1} \Gamma \exp[\pm mu \pm (m^2 - 1)^{1/2}\, v] \tag{85}$$

in which Γ is a constant. Note that all sign combinations are possible. Using (77) and the $(+, -)$ choice of signs, we obtain

$$\varphi(x, t) = \frac{4}{b} \tan^{-1}(\Gamma \exp[\sqrt{ab}\gamma (v_0)(x - v_0 t)]) \tag{86}$$

This solution describes a soliton (kink) moving with a uniform velocity $v_0[\gamma(v_0) = (1 - v_0^2/c^2)^{-1/2}]$. Note that $b\varphi \to O(2\pi)$ as $x \to -\infty$ $(+\infty)$. The field jumps from one of its degenerate vacua to another over a certain region of space.

From a field-theoretic point of view, the sine-Gordon equation can be derived from the following least action principle:

$$\delta \int \mathscr{L}\, d^2x = 0 \qquad \text{where} \quad \mathscr{L} = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - V(\varphi) \tag{87}$$

$$V(\varphi) = \frac{a}{b}(1 - \cos b\varphi) \tag{88}$$

The corresponding energy-momentum tensor is

$$T^{\mu\nu} = \partial^\mu \varphi \partial^\nu \varphi - g^{\mu\nu} \mathscr{L} \tag{89}$$

which yields the following Hamiltonian density:

$$\mathscr{H} = T^{00} = \frac{1}{2c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + V(\varphi) \tag{90}$$

In these equations, $g_{\mu\nu} = \text{diag}(+1, -1)$ is the Minkowski metric tensor in 1 + 1 dimensions, and $x^\mu = (ct, x)$. The following topological current can be defined (Rajaraman, 1982):

$$J^\mu = \frac{b}{2\pi} \varepsilon^{\mu\nu} \partial_\nu \varphi \tag{91}$$

This current is conserved ($\partial_\mu J^\mu = 0$) and the corresponding charge is quantized:

$$Q = \int J_0\, dx = \frac{b}{2\pi}(\varphi(+\infty) - \varphi(-\infty)) = n, \qquad n \in Z \tag{92}$$

Because the field φ assumes its vacuum values $\varphi_n = 2n\pi b$ at either infinity. The topological charge corresponding to kinks (antikinks) is +1 (−1), respectively.

The corresponding total energy and momentum of the kink are

$$E = \int \mathscr{H} dx = \gamma mc^2 \tag{93}$$

$$P = \int T^{01} dx = \gamma mv \tag{94}$$

where the kink rest mass is

$$m \equiv \frac{1}{c^2} E(v = 0) = 8 \frac{a^{1/2}}{c^2 b^{3/2}} \tag{95}$$

Einstein's equation $E^2 = p^2c^2 + m^2c^4$ is obviously satisfied. In this respect and many other respects to be discussed, the kink behaves like a classical relativistic point particle, although it is an extended object and has wave nature inherent in it.

4.3. Double-Kink Solutions

Let $k = 0$, $m > 1$, and $n \neq 0$. In this case,

$$U(u) = \pm\frac{n}{m} \sinh(mu + c_1)$$

$$V(v) = \frac{n}{\sqrt{m^2-1}} \cosh\left(\sqrt{m^2 - 1}v + c_2\right)$$

Adopting a suitable origin for u and v coordinates, the corresponding solution becomes

$$\varphi(x, t) = \pm \frac{4}{b} \tan^{-1}\left[\sqrt{1 - 1/m^2} \frac{\sinh(\sqrt{abm}x)}{\cosh(\sqrt{ab(m^2 - 1)}ct)}\right] \tag{96}$$

The topological charge for this solution is

$$Q = \frac{b}{2\pi} (\varphi(+\infty) - \varphi(-\infty)) = \frac{b}{2\pi} \left(\pm\frac{2\pi}{b} \pm \frac{2\pi}{b}\right) = \pm 2 \tag{97}$$

which clearly explains the double-kink nomenclature. At $t \to -\infty$ and $x \to +\infty$,

$$\varphi(x, t) \simeq \pm\frac{4}{b} \tan^{-1}\left[\sqrt{1 - 1/m^2} \exp\left(\sqrt{abm}\left(x + \sqrt{1 - 1/m^2}ct\right)\right)\right]$$

which describes an antikink (kink) moving in the $-x$ direction with an initial velocity $v_o = \sqrt{1 - 1/m^2}c$. There is also a kink (antikink) at $x \to -\infty$ moving in the $+x$ direction at the same initial velocity. These two kinks move toward each other, collide at $t = 0$, and then recede from each other for $t > 0$ (see Fig. 1). We introduce an intuitive definition for the kink's position, by twice differentiating (96) with respect to x, and requiring $\varphi_{xx} = 0$. For the present case this yields

$$x_k(t) = \pm\frac{1}{\sqrt{ab}\gamma} \cosh^{-1} \sqrt{\frac{1}{\beta^2} \cosh^2\left(\sqrt{ab}\gamma\beta ct\right) - 1} \tag{98}$$

In this equation $\beta = v_o/c$ and $\gamma = (1 - \beta^2)^{-1/2}$.

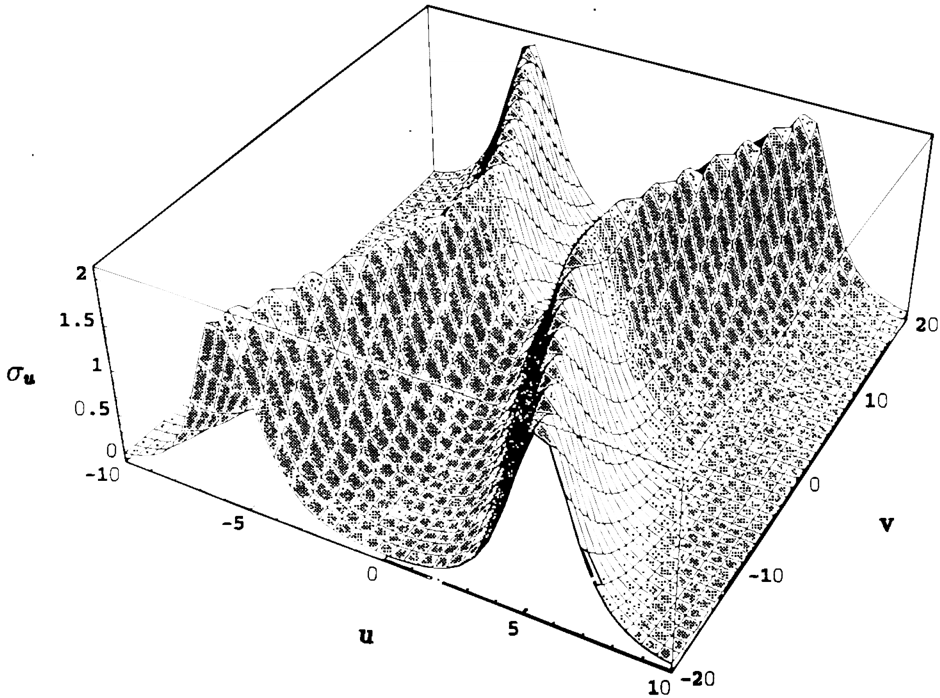

Fig. 1. *k–k* collision.

It can be noted that $x_k \to \pm v_0 t$ as $t \to \pm\infty$. This corresponds to long before and long after collision at $t = 0$, i.e., when there is no interaction and the kinks move with uniform velocities.

A velocity-dependent force can be extracted from (98) and its first and second derivatives ($\dot{x}_k \equiv v_k$ and $\ddot{x}_k \equiv a_k$):

$$\frac{v_k(t)}{v_0} = \frac{1}{\beta^2} \frac{\sinh(2\sqrt{ab}\gamma v_0 t)}{\sinh(\sqrt{ab} x_k(t))} \tag{99}$$

$$a_k(t) = 2\sqrt{ab}\gamma v_0 v_k(t) \times \left[\coth(2\sqrt{ab}\gamma v_0 t) - \frac{v_k(t)}{v_0}\coth(2\sqrt{ab}\gamma x_k(t))\right] \tag{100}$$

Equations (98) and (99) can, at least numerically, yield $t(x_k, v_k)$ and $v_0(x_k, v_k)$. Equation (100) will then express a velocity-dependent acceleration as a function of the interkink separation $2x_k$ and the kink velocity v_k. Figure 2 shows this acceleration as a function of x_k for several values of the initial velocity v_0.

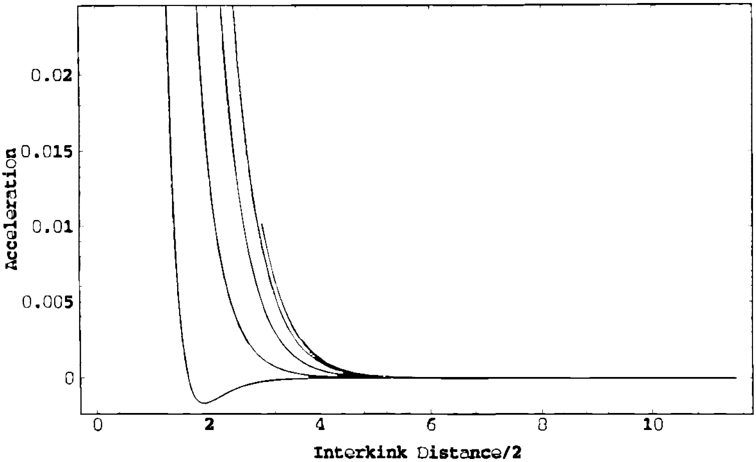

Fig. 2. Kink acceleration as a function of kink position for various initial velocities ($k + k \to k + k$).

4.4 Kink–Antikink Solution

This solution corresponds to $k \neq 0$ and $n = 0$ in (83) and (84). For $m^2 > 1$, we have

$$\varphi(x, t) = -\frac{4}{b}\tan^{-1}\left[\frac{m}{\sqrt{m^2 - 1}}\,\frac{\sinh(\sqrt{m^2 - 1})abct)}{\cosh(m\sqrt{ab}x)}\right] \tag{101}$$

which describes the collision of an unbound kink–antikink pair moving initially with velocities $v_0 = c(1 - 1/m^2)^{1/2}$.

If $m = 1$, the following solution is obtained:

$$\varphi(x, t) = -\frac{4}{b}\tan^{-1}[\sqrt{ab}ct\ \mathrm{sech}\sqrt{ab}x] \tag{102}$$

Note that the total charge is zero for both (101) and (102). Equation (102) describes the limiting case of the kink–antikink relative velocity becoming zero at infinity. In this case

$$x_k(t) = \pm\frac{1}{\sqrt{ab}}\sinh^{-1}\sqrt{1 + abc^2t^2} \tag{103}$$

The two solitons have a closest separation

$$d_{\min} = 2x_k(t = 0) = \frac{2\ln(1 + \sqrt{2})}{\sqrt{ab}} \tag{104}$$

The kink velocity ($v_k = \dot{x}_m$) is obtained from (103)

$$\frac{v_k}{c} = \frac{\sqrt{abct}}{\sqrt{(1 + abc^2t^2)(2 + abc^2t^2)}} \tag{105}$$

which is zero at $t = 0$ and $t \to \pm\infty$. It is interesting to note that although the pair never get closer than $d_{\min}$, there is a charge transfer at this distance, and for a macroscopic observer it looks as if the two solitons are penetrating each other.

The kink acceleration is obtained as

$$a_k(t) = \frac{v_k}{t}\left[1 - \frac{v_k^2}{c^2}(6 + 4abc^2t^2)\right] \tag{106}$$

which is plotted as a function of x_k in Fig. 3. It can be seen that the acceleration changes sign at

$$x_1 = \frac{1}{\sqrt{ab}} \sinh^{-1}(\sqrt{1 + \alpha^2}) \tag{107}$$

where $\alpha = [(\sqrt{33} - 3)/2]^{1/2}$.

The form of potential is interesting and resembles physical potentials like the nuclear and van der Waals potentials.

4.5. Breather Solution

This solution corresponds to $k \neq 0$, $n = 0$, and $m^2 < 1$:

$$\varphi(x, t) = \pm\frac{4}{b}\tan^{-1}\left[\frac{m}{\sqrt{1 - m^2}} \frac{\sin(\sqrt{(1 - m^2)abct})}{\cosh(m\sqrt{abx})}\right] \tag{108}$$

It describes a kink–antikink bound state (see Fig. 4). It can be shown that in this case

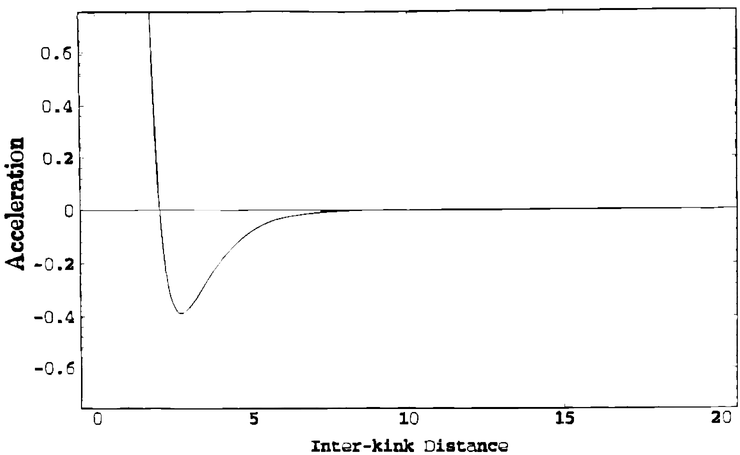

Fig. 3. Kink acceleration as a function of kink position for the critically bound k–$\tilde{k}$ pair.

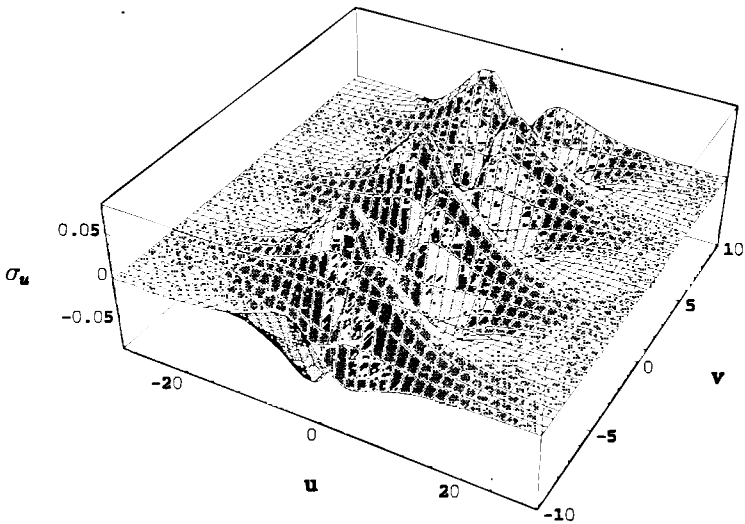

Fig. 4. Breather solution.

$$x_k(t) = \pm\frac{1}{m\sqrt{ab}}\sinh^{-1}\sqrt{1+\frac{m^2}{1-m^2}\sin^2\sqrt{1-m^2)abct}} \tag{109}$$

which clearly shows the periodic nature of the solution with a period

$$\omega_B = c\sqrt{(1-m^2)ab} \tag{110}$$

The maximum distance between the pair is

$$x_{\max} = \frac{2}{m\sqrt{ab}}\sinh^{-1}\frac{1}{\sqrt{1-m^2}} \tag{111}$$

It is seen that $x_{\max} \to \infty$ and $\varphi_B \to 0$ as $m \to 1^-$. The kink velocity can be expressed as

$$\frac{v_k(t)}{c} = \frac{m}{\sqrt{1-m^2}}\frac{\sin 2\sqrt{(1-m^2)abct}}{\sinh 2m\sqrt{ab}x_k} \tag{112}$$

Note that the case $k = 0$, $m < 1$, and $n \neq 0$ is essentially equivalent to the breather solution, since a redefinition $n'^2 = -n^2$ gives a solution

$$\varphi(x, t) = \pm\frac{4}{b}\tan^{-1}\left[\frac{\sqrt{1-m^2}}{m}\frac{\cosh m\sqrt{ab}x}{\sin\sqrt{(1-m^2)abct}}\right] \tag{113}$$

which is nothing but a 2π shift in the φ of the breather solution (108).

The corresponding acceleration is calculated numerically and plotted in Fig. 5 for various values of the parameter m. The form of these curves suggests that the notion of a classical binding "potential" between the kink and antikink is plausible only in the $m \to 1$ limit. In this limit, the form of potential approaches that of Fig. 3.

5. SOLUTIONS GENERATED BY BÄCKLUND TRANSFORMATIONS

A description of the Bäcklund transformation and its use to obtain multisoliton solutions of nonlinear equations can be found in Lamb (1980). Here we present a short discussion of the transformations, restricting ourselves to the sine-Gordon equation. The basic idea behind the Bäcklund transformations is the following. Although we are concerned with nonlinear equations in which superposition of solutions is no longer valid as a solution, multisoliton solutions could be generated by algebraic means, which in some respect resembles the superposition procedure used for linear equations.

The sine-Gordon equation (78) can be written in the form

$$\sigma_{\varepsilon\tau} = \sin\sigma \tag{114}$$

using the change of variables

$$\xi = \tfrac{1}{2}(u - v), \qquad \tau = \tfrac{1}{2}(u + v) \tag{115}$$

It can be shown that (114) is invariant under the following transformations:

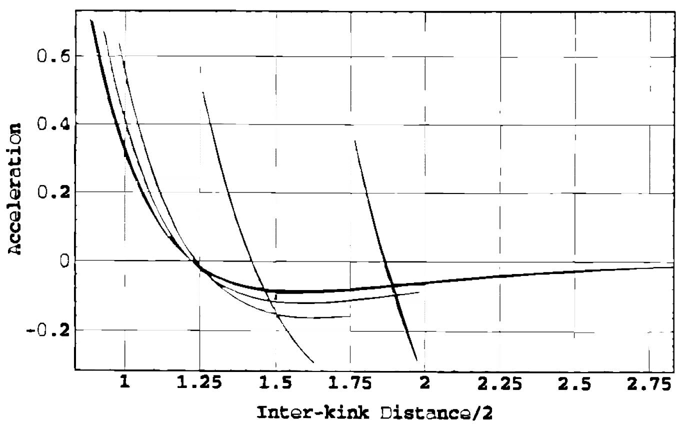

Fig. 5. The k–$\tilde{k}$ acceleration as a function of x_k for various values of m.

$$\sigma'_\xi = \sigma_\xi - 2\beta \sin\left(\frac{\sigma\sigma^1}{2}\right) \tag{116}$$

$$\sigma'_\tau = -\sigma_\tau + \frac{2}{\beta} \sin\left(\frac{\sigma - \sigma^1}{2}\right) \tag{117}$$

$$\sigma' = \sigma; \qquad \tau' = \tau \tag{118}$$

where β is a real constant called the Bäcklund parameter. Equivalence of (116) and (117) with the sine-Gordon equation (114) is readily confirmed via the integrability condition

$$2\sigma_{\xi\tau} = \beta(\sigma_\tau + \sigma'_\tau) \cos\left(\frac{\sigma + \sigma'}{2}\right) + \frac{1}{\beta} (\sigma_\xi - \sigma'_\xi) \cos\left(\frac{\sigma - \sigma^1}{2}\right) \tag{119}$$

Using equations (116) and (117), this yields (114).

The inverse Bäcklund transformations

$$\sigma_\xi = \sigma'_\xi + 2\beta \sin\left(\frac{\sigma + \sigma'}{2}\right) \equiv B_1(\sigma', \sigma'_\xi; \sigma) \tag{120}$$

$$\sigma_\tau = -\sigma'_\tau + \frac{2}{\beta} \sin\left(\frac{\sigma - \sigma'}{2}\right) \equiv B_2(\sigma', \sigma'_\tau; \sigma) \tag{121}$$

yield

$$\sigma'_{\xi\tau} = \sin \sigma' \tag{122}$$

which confirms the invariance of the sine-Gordon equation under Bäcklund transformations. Thus, if one has a first solution σ for (114), another solution σ' for (122) can be found by integrating (116) and (117).

As an example, $\sigma = 0$, which is a trivial solution, yields

$$\sigma'_\varepsilon = -2\beta \sin \frac{\sigma'}{2} \tag{123}$$

$$\sigma'_\tau = -\frac{2}{\beta} \sin \frac{\sigma'}{2} \tag{124}$$

These differential equations immediately yield

$$\sigma'(\xi, \tau) = 4 \tan^{-1} e^{-\beta\xi - \beta^{-1}\tau + \alpha} \tag{125}$$

The following theorem (Rogers, 1990) enables us to find multisoliton solutions without executing further integrations:

Permutability Theorem. If σ_{n-1} is solution of (114), and σ'_n and σ''_n are two solutions generated from σ_{n-1} according to (116) and (117) with parameters β' and β'', respectively, there exists a solution σ_{n+1} such that

$$\sigma_{n+1} = 4\tan^{-1}\left[\frac{\beta' + \beta''}{\beta' - \beta''}\tan\left(\frac{\sigma'_n - \sigma''_n}{4}\right)\right] + \sigma_{n-1} \tag{126}$$

The algorithm for generating σ_{n+1} is shown schematically in Fig. 6. This is called a Bianchi diagram.

Note that the Bäcklund parameters β_1 and β_2 are interchanged in this diagram.

Bianchi diagrams for the third- and fourth-generation solutions are shown in Fig 7. and 8, respectively.

Starting from the trivial solution $\sigma_o = 0$ and two first-generation solutions

$$\sigma_1^{(1)} = 4\tan^{-1}\exp(\beta_1\zeta + \beta_1^{-1}\tau + \alpha_1)$$

$$\alpha_1^{(2)} = 4\tan^{-1}\exp(\beta_2\zeta + \beta_2^{-1}\tau + \alpha_2)$$

equation (126) yields the second-generation (double-kink) solution

$$\sigma_2 = 4\tan^{-1}\left[\frac{\beta_1 + \beta_2}{\beta_1 - \beta_2}\tan\left(\frac{\sigma_1^{(1)} - \sigma_1^{(2)}}{4}\right)\right] \tag{127}$$

and the third-generation (three-kink) solution

$$\sigma_3 = 4\tan^{-1}\left[\frac{\beta_1 + \beta_3}{\beta_1 - \beta_3}\tan\left(\frac{\sigma_2^{(1)} - \sigma_2^{(2)}}{4}\right)\right] + \sigma_1^{(2)} \tag{128}$$

where

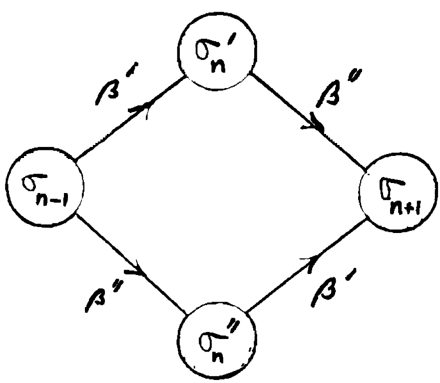

Fig. 6. Bianchi diagram for the permutability theorem.

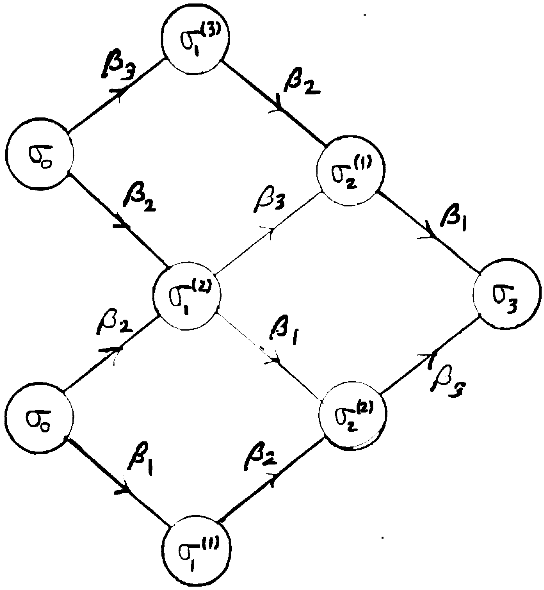

Fig. 7. Third-generation Bianchi diagram.

$$\sigma_2^{(1)} = 4\tan^{-1}\left[\frac{\beta_1+\beta_2}{\beta_1-\beta_2}\tan\left(\frac{\sigma_1^{(1)}-\sigma_1^{(2)}}{4}\right)\right] \tag{129}$$

$$\sigma_2^{(2)} = 4\tan^{-1}\left[\frac{\beta_2+\beta_3}{\beta_2-\beta_3}\tan\left(\frac{\sigma_1^{(2)}-\sigma_1^{(3)}}{4}\right)\right] \tag{130}$$

and

$$\sigma_1^{(i)} = 4\tan^{-1} e^{\beta_i\xi+\beta_i^{-1}\tau+\alpha_i}; \qquad i = 1, 2, 3 \tag{131}$$

It can be easily shown that the topological charge of the solution (127) is 0, ± 2, corresponding to $k\tilde{k}$, kk, and $\tilde{k}\tilde{k}$; combinations. The topological charge of (128) is ± 1, ± 3, corresponding to kkk, $\tilde{k}\tilde{k}\tilde{k}$, $kk\tilde{k}$, and $\tilde{k}\tilde{k}k$ combinations.

Note that although (127) represents two kinks moving always in the positive ξ direction, we can always find boosts which make kinks move in either direction.

In order to confirm the conformity of the two-kink solution (127) with the previously derived solutions of Section 4, we apply the following Lorentz transformations:

$$x = \gamma(x' + ut'); \qquad t = \gamma(t' + ux'/c^2) \tag{132}$$

We therefore have

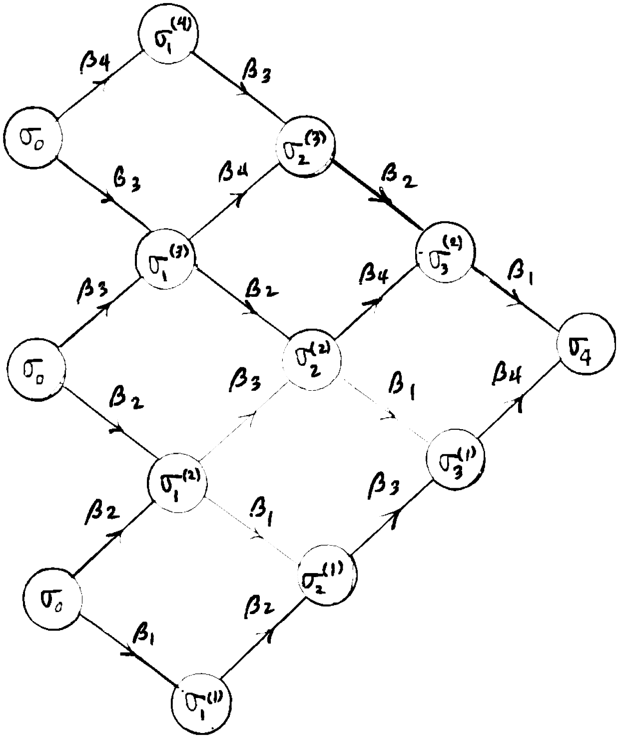

Fig. 8. Fourth-generat ion Bianchi diagram.

$$\varphi_1^{(i)} = \frac{4}{b}\tan^{-1}\exp\left[\sqrt{ab}\gamma[(\beta_i + \beta_i^{-1}\,u/c)x' + (\beta_1 u+\beta_i^{-1}\,c)t']\right]; \qquad i = 1, 2 \tag{133}$$

We need the two kinks to have equal and opposite velocities in the new frame (x', t'). Therefore

$$\frac{\beta_1 u + \beta_1^{-1}c}{\beta_1 + \beta_1^{-1}u/c} = -\frac{\beta_2 u + \beta_2^{-1}c}{\beta_2 + \beta_2^{-1}u/c} \tag{134}$$

Now, choosing the parameters β_1 and β_2 according to

$$m = \gamma(\beta_1 + \beta_1^{-1}u/c) = -\gamma(\beta_2 + \beta_2^{-1}u/c) \tag{135}$$

and doing a little algebra reveals that (127) yields (96).

6. KINK DYNAMICS IN AN INHOMOGENEOUS MEDIUM

In this section we examine the dynamics of sine-Gordon kinks in an inhomogeneous medium. The time-independent inhomogeneity can be introduced into the sine-Gordon equation in different ways. We will mention three ways, although only two cases will be worked out in detail.

The first kind of inhomogeneity is introduced via a varying refractive index':

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{n^2(x)}{c^2}\frac{\partial^2\varphi}{\partial t^2} = a \sin b\varphi \tag{136}$$

This looks like a variation in the optical refractive index of a transparent medium in the context of electromagnetic wave propagation. A physical fulfillment of equation (136) can be accomplished by an inhomogeneous Josephson transmission line.

Consider equation (22). If the dielectric constant K is to vary with x, we can write

$$c_J = \frac{\tilde{c}}{n(x)} \tag{137}$$

where

$$\tilde{c} = \frac{1}{\sqrt{LC_0}} \tag{138}$$

and

$$n(x) = \sqrt{\frac{K(x)}{K_0}} \tag{139}$$

In equation (138), C_0 is a constant reference capacitance corresponding to the dielectric constant K_0.

The inhomogeneity of the second kind can be introduced via spatially varying a and b:

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = a(x) \sin b(x)\varphi \tag{140}$$

This kind of inhomogeneity will be discussed in detail in Section 6.2.

The third kind of inhomogeneity is introduced via an external field $\chi(x)$:

$$\frac{\tau^2\varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = a \sin b\varphi + \chi(x) \tag{141}$$

This case is discussed in Reinisch and Fernandez (1981) and Kaup (1984). It seems less interesting to us, and therefore we shall not consider it any further.

Several interesting questions arise when the kink dynamics in an inhomogeneous medium is concerned. If the kink is considered as a classical particle interacting with a background potential, what would be the characteristics of such a potential in relation with $n(x)$, $a(x)$, $b(x)$, and the kink velocity v_k? In what circumstances does the particle aspect fail to be adequate? What are the interesting features of the wave aspect? etc.

Although interesting analytical approximations can be worked out which explain some of the basic properties of the kink dynamics, a more elaborate picture of what is going on can only be achieved via numerical integration.

The numerical procedure we have followed in order to carry out the integration is as follows. The x axis in the relevant range is divided into N divisions of length ε. The time axis is also divided into intervals of duration δ. Using the standard finite-difference expressions for spatial and temporal derivatives, we can show that

$$\sigma_{i,j+1} = \frac{\delta^2(\sigma_{i+1,j} - 2\sigma_{i,j} + \sigma_{i-1,j} - \varepsilon^2 a_i \sin b_i \sigma_{i,j})}{\varepsilon^2 n_i^2} + 2\sigma_{i,j} - \sigma_{i,j-1} \tag{142}$$

where $\sigma_{i,j} \equiv \sigma(u = i\varepsilon, v = j\delta)$, $a_i = a(u = i\varepsilon)/a_0$, $b_i = b(u = i\varepsilon)/b_0$, and $n_i = n(u = i\varepsilon)$. As before, $\sigma = b_0\varphi$, $u = \sqrt{a_0 b_0}x$, and $v = \sqrt{a_0 b_0}ct$, a_0 and b_0 being some reference values for a and b, respectively.

Equation (142) enables us to calculate the field configuration in a next time step, using its configuration at two preceding time steps. Starting from an initial configuration at time steps $j = 1$ and $j = 2$, equation (142) can be successively applied to calculate the kink dynamics up to any arbitrary later time. One, however, should be careful about the numerical instabilities which may arise. See, for example, Chapter 17 of Press *et al.* (1986), and in particular the corresponding sections on von Neumann stability analysis and the Lax method.

6.1. Inhomogeneity of the First Kind

Kink dynamics in a medium with $n(x)$ can be approximated in terms of a classical particle moving against a *velocity-dependent* potential. Consider a kink with initial velocity v_0 incident on a 'potential barrier'

$$n(x) = \begin{cases} 1.0 & x < x1 \\ n1 & x > 1 \end{cases} \tag{143}$$

Numerical results indicate that low-velocity kinks *do* penetrate the barrier. There exists a threshold velocity, above which kinks *cannot* penetrate the barrier. This threshold velocity depends, of course, on the potential height n_1. Ultrarelativistic kinks cause kink–antikink pair production together with low-amplitude excitations. Examples are shown in Fig. 9 for $n_1 = 1.02$ and $u_k = 0.01$, 0.1, and 0.97.

The velocity dependence of the force acting on the kink is evident also in Fig. 10, where a kink moves across a linearly increasing refractive index

$$n(x) = \begin{cases} 1.0 & x < x1 \\ 1 + a(x - x1) & x \geq x1 \end{cases} \tag{144}$$

The kink is observed to penetrate to a certain depth, where it is almost frozen. It will never come back to recover its 'potential energy.'

A simple analytical description can be presented in the limit of slowly varying refractive index and over short periods of time. Consider the coordinate transformations.

$$\tilde{x} = x \tag{145}$$

$$\tilde{t} = \frac{t}{n(x)} \tag{146}$$

under which the sine-Gordon equation becomes

$$\frac{\partial^2 \varphi}{\partial \tilde{x}^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial \tilde{t}^2} + \tilde{t}\frac{n'}{n}\left[-2\frac{\partial^2}{\partial \tilde{t}\partial \tilde{x}} + \frac{n'}{n}\frac{\partial}{\partial \tilde{t}} + \tilde{t}\frac{\partial^2}{\partial \tilde{t}^2}\right]\varphi = a \sin b\varphi \tag{147}$$

in which $n' = dn/dx$. For the kink solution, $\partial^2\varphi/\partial x^2 = a.b.O(1)$ and $1/c^2$ $\partial^2\varphi/\partial t^2 = a.b.O(v_k^2/c^2)$. In the limit $|\tilde{t}n'/n| << v_k/c^2$, we can ignore the bracket terms in (147), and obtain the sine-Gordon equation in the $(\tilde{x}, \tilde{t})$ coordinates. The kink solution in the new coordinates reads

$$\varphi(\tilde{x}, \tilde{t}) = \varphi(x, t) = 4 \tan^{-1} \exp[\sqrt{ab}\gamma(v_0)(\tilde{x} - v_0\tilde{t})] \tag{148}$$

in which v_0 is constant. Note that in the (x, t) coordinates $\tilde{x} - v_0\tilde{t} = x - [v_0/n(x)]t$, which corresponds to a varying kink velocity. The instantaneous position of the kink is obtained by solving the equation

$$n(x_k)x_k = v_0 t \tag{149}$$

for $x_k(t)$. The instantaneous acceleration of the kink is

$$a_k(x_k, v_k) = -\frac{2n'(x_k) + n''(x_k)x_k}{(n(x_k) + n'(x_k)x_k)^2} v_0 v_k \tag{150}$$

The potential acting on the kink as a result of varying refractive index of the medium is clearly velocity dependent. The velocity dependence is linear in this approximation. This is in agreement with the previous numerical results.

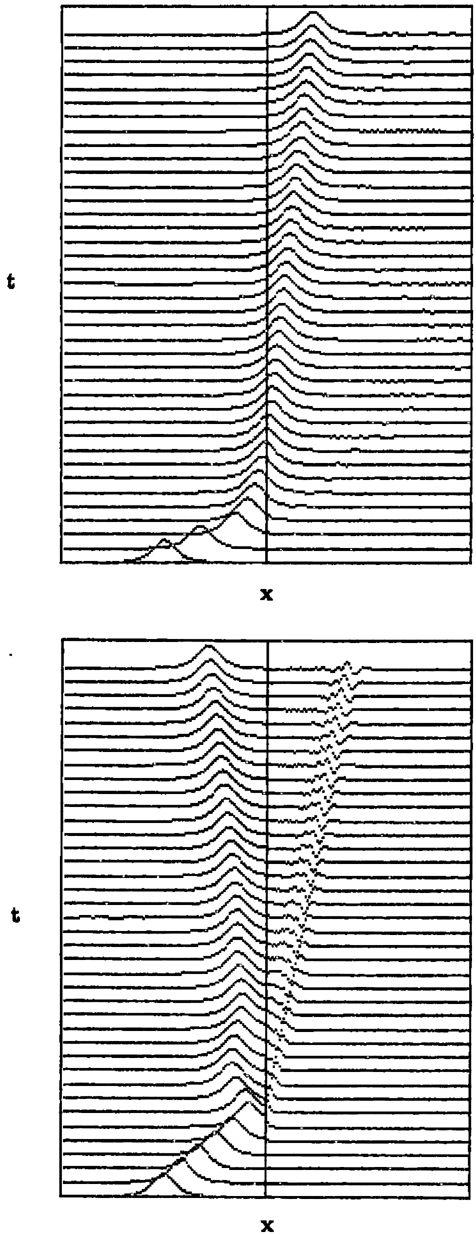

Fig. 9. Kink colliding with a potential step. (a) For $v_k = 0.01$ kink crosses over; (b) for $v_k = 0.1$ kink is reflected.

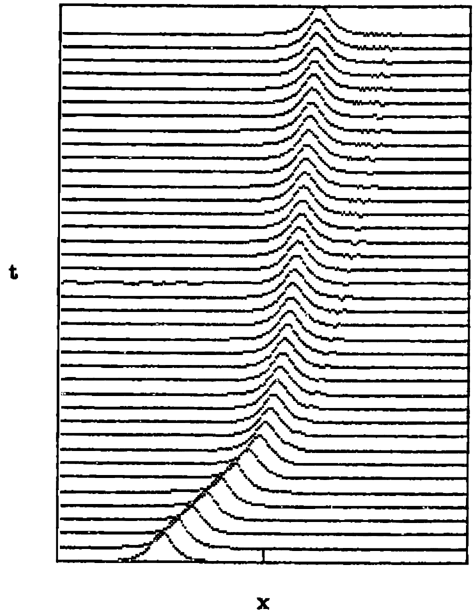

Fig. 10. Kink moving across a linearly increasing refractive index.

6.2. Inhomogeneity of the Second Kind

This kind of inhomogeneity seems to be more interesting and viable to analytical description.

Equation (140) with a and b constant possesses kink solutions having rest energy

$$E_k(0) = 8\,\frac{a^{1/2}}{b^{3/2}} \tag{151}$$

and total moving energy

$$E_k(v_k) = \gamma(v_k)E_k(0) \tag{152}$$

Let us define a reference rest energy E_0 according to

$$E_0 = 8\,\frac{a_0^{1/2}}{b_0^{3/2}} \tag{153}$$

If the x coordinate is now divided into segments with various (but constant) a's and b's, (152) can be written in the following form:

$$E_k(v_k) = E_0 + T + U \tag{154}$$

where

$$U = E_k(0) - E_0 \tag{155}$$

That is, we have naturally decomposed the total kink energy in a locally homogeneous medium into three parts: rest energy E_0, kinetic energy T, and potential energy U. From (151), (153), and (155), we have

$$U = 8\frac{a^{1/2}}{b_{3/2}} - 8\frac{a_0^{1/2}}{b_0^{3/2}} = E_0\left[\sqrt{\frac{a}{a_0}}\left(\frac{b_0}{b}\right)^{3/2} - 1\right] \tag{156}$$

Let us now tentatively generalize this definition to the case $a = a(x)$ and $b = b(x)$. The kink dynamics can then be described through the conventional prescription of classical relativistic dynamics, *as long as the scale over which the potential varies appreciably is large compared with the size of the kink.* The force field turns out to be nearly conservative, and common energy-conservation arguments for a massive particle approximately hold.

Figure 11 shows the example of a kink moving against a potential slope. The kink returns to its initial position and velocity after a finite time.

As a classical particle, the kink bypasses a potential barrier as long as its kinetic energy is more than the potential height (see Fig. 12). However, it is interesting to note that it *does* penetrate a potential barrier with $U > T$ if the barrier is thin enough. Figure 13 shows this classical 'tunneling effect.' This effect is a result of the wave aspect of the kink and does not contradict

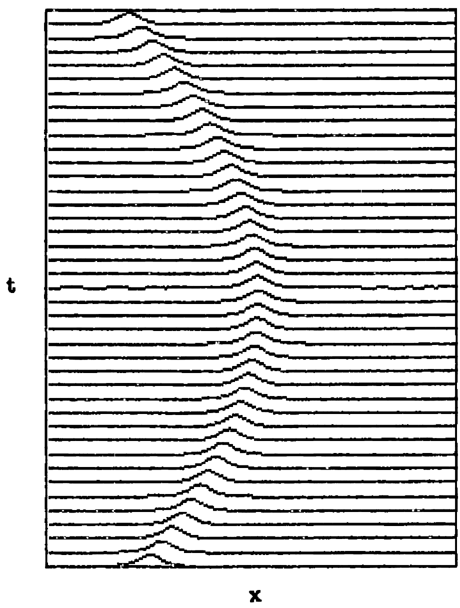

Fig. 11. Kink moving across a potential slope (second kind).

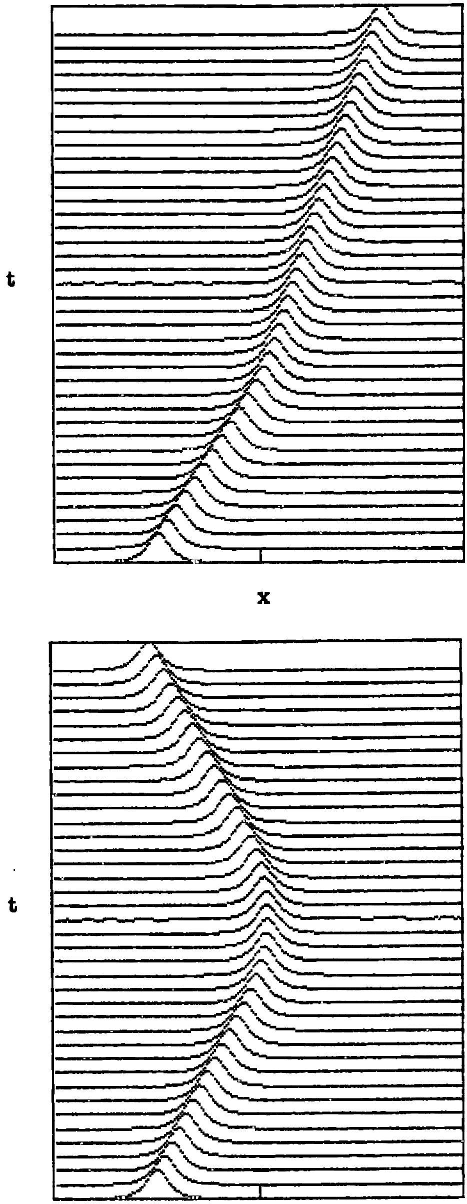

Fig. 12. Kink passing over a potential step for $T > U$ (a), and being reflected for $T < U$ (b).

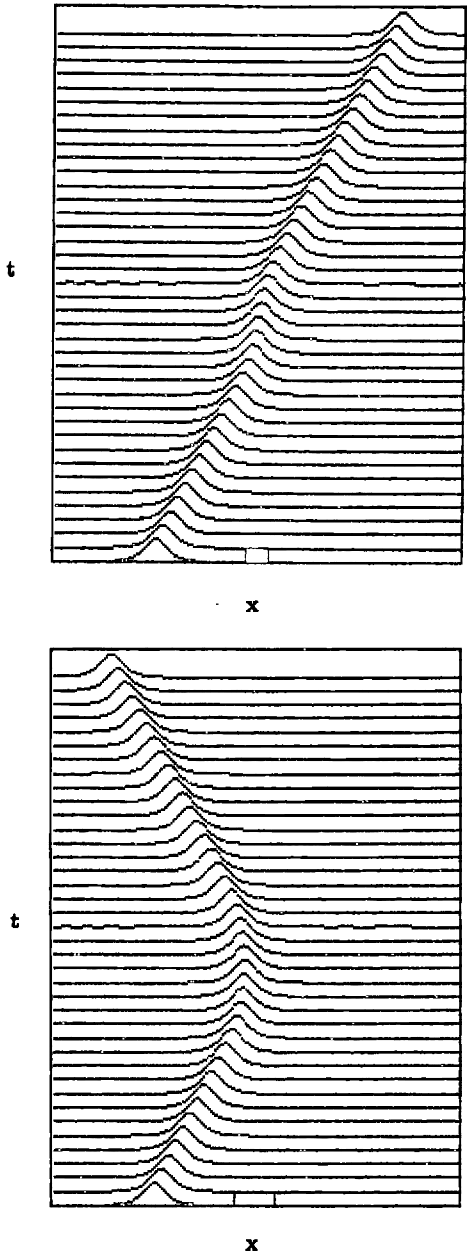

Fig. 13. (a) Classical tunneling of a kink through a thin barrier with $T < U$. (b) For a wider barrier (and the same kinetic energy) there is no tunneling.

the particle aspect just described, because in this case the slowly varying assumption for the potential breaks down.

6.3. The Mass of Potential Energy

The analytical prescription put forward in Section 6.2 raises an interesting question about the mass of potential energy. This fundamental problem was first discussed in detail by Brillouin (1965a, 1965b), who wrote:

> Every scientist writes $E = Mc^2$, but almost everybody forgets to use this relation for potential energy. The founders of relativity seemed to ignore the question, although they specified that (this) relation must apply to all kinds of energy, mechanical, chemical, etc. When it comes to mechanical problems, the formulas usually written contain the mass of kinetic energy, but they keep silent about the mass of potential energy.

The problem of potential energy mass can be easily clarified in the context of kink dynamics: Equation (155) can be written in the form

$$E_k(v_k = 0, a \neq a_0, b \neq b_0) = E_0(v_k = 0, a = a_0, b = b_0) + U = m_k c^2 \tag{157}$$

where $m_k = 8a^{1/2}/c^2 b^{3/2}$. What we call 'potential energy' of the kink is nothing but the mass increase (or decrease) as a result of interaction with the medium against which the kink moves. The inertial mass of the kink, therefore, certainly depends on the background values of a and b and the potential energy contributes to its rest mass.

7. GENERALIZED SINE-GORDON EQUATIONS

Single solitons of the sine-Gordon equation are degenerate and have equivalent masses (Rajaraman, 1982). We can change the Lagrangian density of the sine-Gordon equation in such a way that this degeneracy will be removed. The Lagrangian density can be generalized at least in two ways. First we multiply the Lagrangian density by a factor $f(\varphi)$ in such a way as to obtain the desired results. In the second method, we change the potential term (88) so that the parameter a will be a piecewise-constant function of φ. By considering (95), it is obvious that the rest mass of the solitary wave will be changed. We now consider these two methods in more detail.

7.1. Multiplication by a Scalar Function

Let us multiply the Lagrangian density of the sine-Gordon equation by $[\frac{1}{2}(r + 2)]^2\psi^r$:

$$\mathcal{L}(\psi, \partial_\mu\psi) = \left(\frac{r+2}{2}\right)^2 \psi^r \left[\frac{1}{2}\partial^\mu\psi\partial_\mu\psi - (1 - \cos\psi)\right] \tag{158}$$

where r is a real number. This Lagrangian density approaches that of the sine-Gordon equation in the limit $r \to 0$. By changing the variables ψ and r as

$$s = \frac{2}{r+2} \qquad \text{and} \qquad \psi = \phi^s \tag{159}$$

we find that equation (158) becomes

$$\mathcal{L}(\phi, \partial_\mu\phi) = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1 - \cos(\phi)^s}{s^2\phi^{2s-2}} \tag{160}$$

By using the Euler–Lagrange equation in the static case ($\partial\phi/\partial t = 0$) we obtain

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial V(\phi)}{\partial\phi} \tag{161}$$

where

$$V(\phi) = \frac{1 - \cos(\phi)^s}{s^2\phi^{2s-2}} = \frac{2\sin^2(\phi^s/2)}{s^2\phi^{2s-2}} \tag{162}$$

Multiplying both sides of (161) by $\partial\phi/\partial x$ and integrating, we get

$$x - x_0 = \int \frac{d\phi}{\sqrt{2V(\phi)}} = \ln\left(\tan\left(\frac{\phi^s}{4}\right)\right) \tag{163}$$

The exact solution of this static case is thus obtained:

$$\phi(x) = [4\tan^{-1} e^{(x-x_0)}]^{1/s} \tag{164}$$

In order to obtain the time-dependent moving solution, it is enough to express $\phi(x)$ in a x' system which moves with the uniform velocity v with respect to the x system:

$$\phi'(x', t') = \phi(x, t) = [4\tan^{-1} e^{(x' - x'_0)}]^{1/s} \tag{165}$$

If we use

$$x' = \gamma(x - vt), \qquad t' = \gamma(t - vx), \qquad \gamma = \frac{1}{\sqrt{1 - v^2}} \tag{166}$$

we obtain

$$\phi(x, t) = [4\, tan^{-1} e^{\gamma(x - vt - x_0)}]^{1/s} \tag{167}$$

The energy-momentum tensor is given by

$$T^{\mu\nu} = \partial^{\mu}\phi\partial^{\nu}\phi - \eta^{\mu\nu}\mathcal{L} \tag{168}$$

The (o, o) component of energy-momentum tensor yields the Hamiltonian density;

$$T^{o,o} = \mathcal{H} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi) \tag{169}$$

The total energy for the static (164) and dynamic (167) solutions can be obtained accordingly. For the time being, we do the calculation only for the static case:

$$E_0(s, n) = \int_{-\infty}^{\infty} \mathcal{H}\, dx = \int_{(2\pi n)^{1/s}}^{[2\pi(n+1)]^{1/s}} \sqrt{2V(\phi)}\, d\phi \tag{170}$$

where $E_0(s, n)$ is the rest energy of the solitary wave between two vacuum states n and $n + 1$. By considering (162), equation (170) gives

$$E_0(s, n) = \frac{2^{2/s}}{s^2} \int_{\pi n}^{\pi(n+1)} |u^{(2/s)-s}\, sin\, u|\, du \tag{171}$$

where $u = \phi^s/2$. It is obvious that the above equation yields different rest masses for different n's and s's. For specific values of s, $E_0(s, n)$ can be obtained in terms of elementary functions. For example, if $s = 1$, then $E_0(1, n) = 8$, which is the degenerate rest mass of the sine-Gordon soliton, independently of n (see Section 4.2). For $s = 1/2$,

$$E_0(1/2, n) = 64\, |[\pi^2(2n^2+2n+1) - 4]| \tag{172}$$

This obviously depends on n, and by increasing n, $E_0(1/2, n)$ increases, too. For $s = 2/3$ equation (171) becomes

$$E_0(2/3, n) = 18\pi(2n + 1) \tag{173}$$

For $s = 2$ the rest mass becomes

$$E_0(2, n) = \sum_{m=0}^{\infty} \frac{(-1)^m \pi^2 m^{+1}}{(2m + 1)!(2m + 1)} [(n + 1)^{2m+1} - n^{2m+1}] \tag{174}$$

which decreases by increasing n.

It can be observed that for all values of s (exept $s = 1$), the rest mass $E_0(s, n)$ depends on n.

Using the fact that the Lagrangian density is a scalar (invariant under Lorentz trasformations), it can be shown that the total energy of the moving solitary waves satisfies the Einstein relations $E(s, n) = \gamma E_0(s, n)$ and $E^2(s, n) = P^2(s, n) + E_0^2(s, n)$.

Numerical calculations show that the solutions (167) are stable, while a system of a kink and an antikink does not survive collisions, except for $s = 1$, in which case the solitary waves are solitons.

7.2. Stepwise Constant Parameters

In the second approach, we define the potential function $V(\phi)$ as

$$V(\phi) = a(1 - \cos\phi) \tag{175}$$

in which

$$a = \begin{cases} a_1 & \phi < 0 \\ a_2 & \phi \geq 0 \end{cases} \tag{176}$$

where a_1 and a_2 are constants. Using this extension, single-soliton solutions of the dynamical equation (78) are essentially unchanged. However, by considering equation (95), it can be observed that the solitons belonging to the $\phi \geq 0$ and $\phi < 0$ sectors have different masses.

Numerical calculations show that the interaction between two solitons with velocities v_1 and v_2 and masses m_1 and m_2 show interesting phenomena, which are summarized as follows:

I. A soliton–antisoliton pair undergoing collision can exchange momentum and energy. This is in contrast to the usual interaction of solitons, which can at most lead to a phase shift (Fig. 14).

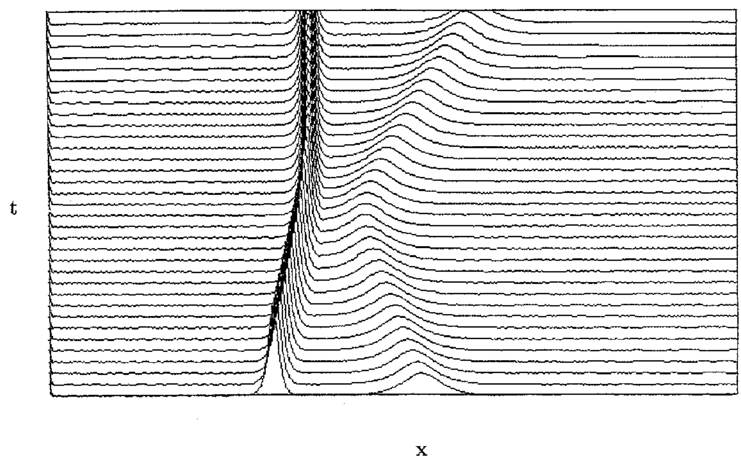

Fig. 14. Momentum transfer between two kinks belonging to different sectors in the modified sine-Gordon (MSG) equation.

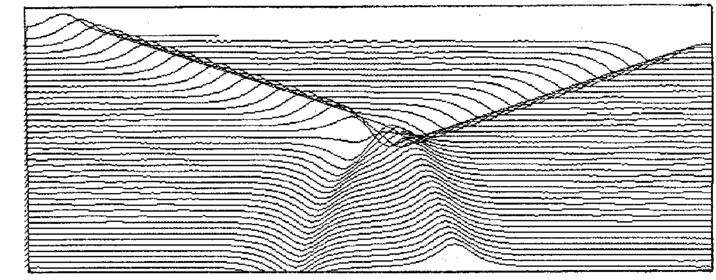

Fig. 15. A heavy kink–antikink pair annihilating to an almost massless pair in the MSG system.

II. A heavy soliton–antisoliton pair colliding at low energies can annihilate and produce a light-weight soliton–antisoliton pair moving at higher velocities (Fig. 15).

III. A low-mass soliton–antisoliton pair colliding at a high enough center-of-mass energy can in a short interval of time produce a bound system of heavy soliton–antisoliton pair and then annihilate to the first pair (Fig. 16).

7.3. Soliton Confinement

Let us add a constant and a harmonic term to the self-interaction potential of the sine-Gordon equation:

$$V(\phi) = 1 + \varepsilon - \cos\phi - \varepsilon\cos(2\phi) \tag{177}$$

where ε is a constant. It is obvious that this potential has false vacuua at $\phi = (2n+1)\pi$ if $\varepsilon > 0.25$.

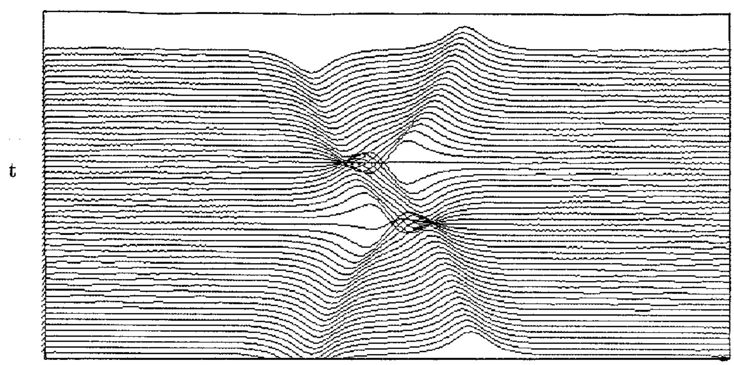

Fig. 16. A low-mass kink–antikink pair produces a transient heavy pair, which subsequently annihilates into the original pair (MSG system).

The dynamical equation corresponding to this potential reads

$$\Box\phi = \sin\phi + 2\,\varepsilon\sin(2\phi) \tag{178}$$

It can be shown that this equation has the exact single-kink (antikink) solution

$$\phi(x, t) = 2\cos^{-1}\left[\pm\frac{\sinh\sqrt{4\varepsilon + 1}\gamma\,(x - vt)}{\sqrt{4\varepsilon + \cosh^2\gamma\sqrt{4\varepsilon + 1}\gamma\,(x - vt)}}\right] \tag{179}$$

in which the $+$ $(-)$ sign is for antikinks (kinks), v is the kink velocity, and $\gamma = (1 - v^2)^{-1/2}$ as before. The corresponding energy density is plotted for $v = 0$ and various values of ε in Fig. 17. The solution (179) becomes the ordinary soliton of the sine-Gordon equation in the limit $\varepsilon \to 0$. As the value of ε grows, the kink splits into two parts (let us call them subkinks). Each of these subkinks has half-integer topological charges and is confined within the kink. A strong force confines them inside the kink, and if energy is supplied to them, they will oscillate back and forth within the kink (Fig. 18). Subkinks do not exist independently.

The strong confining force between the two subkinks can be attributed to the stress tensor of the false vacuum which develops between them as they move apart. If $\phi \simeq (2n + 1)\pi \simeq$ const, then $T^{\mu\nu} \simeq 2g^{\mu\nu}$, which resembles that of a perfect fluid with $p = -\rho = -2$.

The collision between two kinks of this type is shown in Fig. 19, which is rather similar to the collision between two kinks of the conventional sine-Gordon equation. A kink–antikink collision, however, leads to the transfer of some translational kinetic energy into the oscillations of the subkinks.

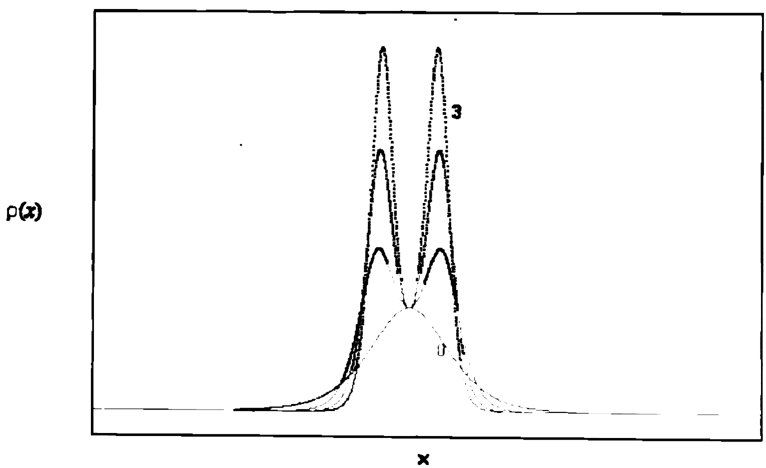

Fig. 17. The energy density of the kink solution in the deformed sine-Gordon system for various values of ε.

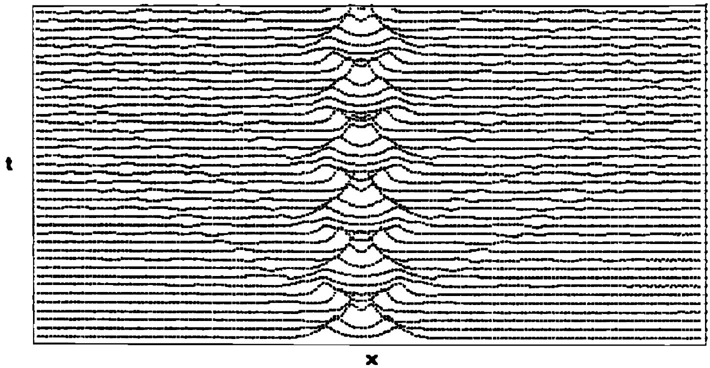

Fig. 18. Excited subkinks oscillate within a kink of the deformed sine-Gordon equation.

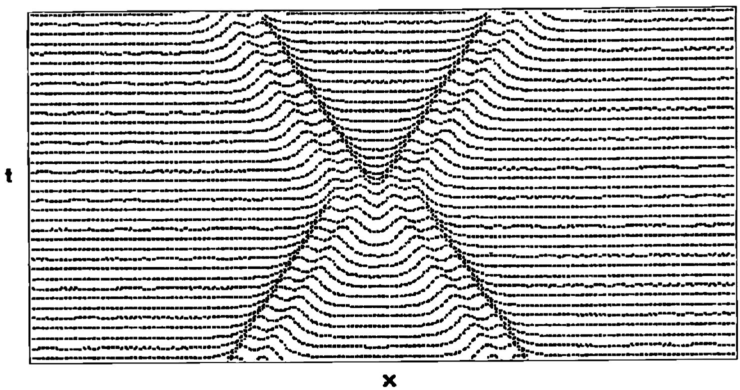

Fig. 19. The collision between two kinks of the deformed sine- Gordon equation ($v_1 = -v_2 = 0.3$).

Once this numerical result is confirmed, one can conclude that the system is not integrable.

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